Probabilistic Laws on Infinite Groups
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Laws on Groups

Definition

A **law** is a non-trivial element $w \in F(x_1, x_2, \dots)$ in the free group.
w is **satisfied** in a group G if $w(g_1, g_2, \dots) = 1$ for any $g_1, g_2, \dots \in G$.

Examples

1. The power law x^k is satisfied in any finite group of order k.
2. The commutator law $[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$ is satisfied in any abelian group.
3. The iterated commutator law $[x_1, [x_2, [\cdots [x_l, x_{l+1}]]$ is satisfied in any l-step nilpotent group.

Can a law be satisfied by 'many' elements but not by all?

A law satisfied by 'many' elements???

Definition

Given a finite (more generally, a compact) group G and a law w, define the probability that w is satisfied in G by:

$$\mathbb{P}(w \text{ holds in } G) := \mathbb{P}_{g_1,\ldots,g_d \sim \mu}(w(g_1,\ldots,g_d) = 1)$$

In many cases, high probability of satisfaction implies global satisfaction.

Fundamental question

Given a group that satisfies a law with high probability, how close is it to satisfying an actual group law?

Example (Gustafson's Theorem)

Let G be a finite group.
If $\mathbb{P}([x_1, x_2] \text{ holds in } G) > 5/8$, then G is abelian.

*Similar results exist for iterated commutator laws, power laws x^k, the metabelian law, and more.

Laws on infinite groups

Let G be a finitely generated, **infinite** group. How to make sense of

$$\mathbb{P}(w \text{ holds in } G)?$$

Balls in the Cayley graph

Fix a symmetric generating set S for G and let $U(n)$ be the uniform measure on the n-ball of the Cayley graph of (G, S). Define:

$$\mathbb{P}(w \text{ holds in } G) := \limsup_{n\to\infty} \mathbb{P}_{U(n)}\left(w(g_1, \dots, g_d) = 1\right)$$

Location of a random walk

Fix a non-degenerate[a] step distribution ν on G. Then ν^{*n} is the n^{th} step distribution with respect to a ν-random walk. Define:

$$\mathbb{P}(w \text{ holds in } G) := \limsup_{n\to\infty} \mathbb{P}_{\nu^{*n}}\left(w(g_1, \dots, g_d) = 1\right)$$

[a] finitely supported distribution, whose support generates G as a semigroup.

Laws on infinite groups

Probabilistic laws on infinite groups were recently studied by Martino, Tointon, Valiunas, and Ventura (nilpotent case); Antol'n, Martino, and Ventura (commutativity); Amir, Blachar, Gerasimova, and Kozma (power laws and more); and others. For instance:

Theorem (Tointon, 2020)

1. *If* $\mathbb{P}([x, y] = 1$ *holds in* $G) > \frac{5}{8}$, *then* G *is abelian.*
2. *If* $\mathbb{P}([x, y] = 1$ *holds in* $G) > 0$ *then* G *is virtually abelian.*

Question (Amir-Blachar-Gerasimova-Kozma)

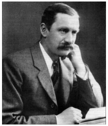
1. *Is there an* $\varepsilon > 0$ *such that if* G *satisfies a power law* $x^k = 1$ *with probability* $> 1 - \epsilon$, *then* G *satisfies* $x^k = 1$*?*
2. *If* G *satisfies a law with probability* 1, *does* G *satisfy that law? a law?*

All groups here are finitely generated, and probabilities are taken with respect to non-degenerate random walk.

Power laws and Burnside groups

The Burnside problem, 1902

- (General.) Is a finitely generated group in which every element has finite order necessarily finite?
- (Bounded.) For which $m, n > 0$ is the free Burnside group $B(m, n) := \langle x_1, \ldots x_m \mid X^n = 1 \text{ for every word } X \rangle$ finite?
- (Restricted.)




Solutions

- Golod-Shafarevich (1964): negative answer to the general problem.
- Novikov-Adian (1968): negative answer to the bounded problem with $n > 4381$ odd.
- Ol'shanskii (1982): concrete construction of $B(m, n)$, $n > 10^{10}$ odd.

Probabilistic Burnside groups via Olshanskii's methods

Theorem (Goffer-Greenfeld, 2023)

There exists a finitely generated $G = \langle S \rangle$, and $k >> 0$ such that:

1. *G satisfies the law $x^k = 1$* **with probability** *1 with respect to $\vec{\mu}_{(G,S)}$;*
2. *G satisfies the law $x^k = 1$* **with probability** *1 with respect to $\vec{\mu}_{(G,\nu)}$ for any finitely supported non-degenerated step distribution ν; and yet*
3. *G* **admits a free subgroup**, *and hence satisfies no group law.*

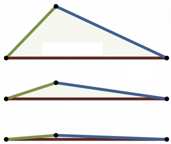
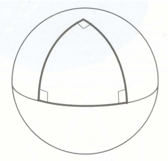
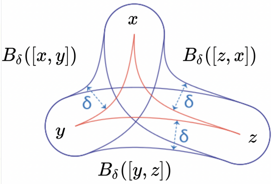
In particular, this answers the two questions of Amir, Blachar, Gerasimova, and Kozma, and provides the first example of a group that satisfies a group law with probability 1 but does not satisfy any group law in full.

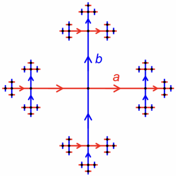
Our method: (hyperbolic groups and) small cancellation theory

Tools: hyperbolic groups

Definition

A metric space is called *hyperbolic* if there exists $\delta > 0$ such that any geodesic triangle is δ-thin.



Definition

A finitely generated group $G = \langle S \rangle$ is called *hyperbolic* if it admits a hyperbolic Cayley graph.

Example

Finite groups, $\mathbb{Z}$, free groups and groups acting on trees are hyperbolic. $\mathbb{Z}^2$ (and any group containing it) is not hyperbolic.

Tools: small cancellation theory

Definition

Let $G = \langle S \mid \mathcal{R} \rangle$, $\mathcal{R}$ symmetrized. A maximal common initial segment u of $R_1, R_2 \in \mathcal{R}$ is called a **piece**. E.g., ab for $R_1 = abc$ and $R_2 = ab^2$.

Definition

$G = \langle S \mid \mathcal{R} \rangle$ is said to satisfy $C'(\lambda)$ **small cancellation condition (scc)** if whenever a subword $u \subset R$ is a piece, then $|u| \leq \lambda |R|$.

Example

1. $\langle a, b, c, d \mid abcbc, abdbd \rangle$ satisfies $C'(2/5)$-scc.
2. $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ satisfies $C'(1/4)$-scc.
   (Here: $\mathcal{R} = aba^{-1}b^{-1}, ba^{-1}b^{-1}a, a^{-1}b^{-1}ab, \dots$)

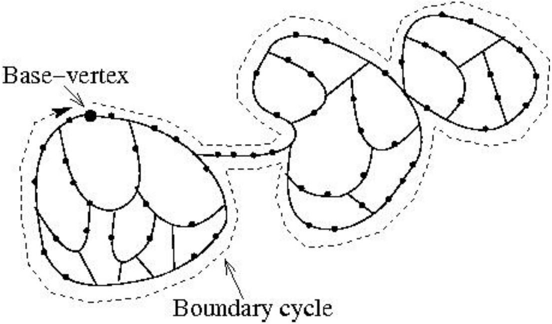
Tools: small cancellation theory

Intuition

In a small-cancellation presentation distinct relations have small overlap.

Lemma

If a finite presentation $G = \langle S \mid \mathcal{R} \rangle$ is $C'(\lambda)$, with $\lambda < 1/6$ then G is non-elementary hyperbolic.



Tools: small cancellation theory

The approach

Name a property you wish the elements of your group to have.

1. Start with a free group $G_0 = F(a, b, c)$.
   Enumerate its elements: $g_1, g_2, \ldots$
2. On the nth step add a small cancellation relation to force g_n to have your desired property. Set $G_n = \langle a, b, c \mid R_1, R_2, \ldots, R_n \rangle$.
3. At the limit group, $G = \langle a, b, c \mid R_1, R_2, \ldots \rangle$, all elements have that property.

Examples

- Olshanskii's solution to Burnside's problem.
- Osin's construction of an infinite group in which all non-trivial elements are conjugate.
- Goffer-Lazarovich's solution for Wiegold's question from the 70's.